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Epstein–Glaser renormalization and differential renormalization

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Abstract. In the framework of causal perturbation theory by Epstein and Glaser the process of renormalization is precisely equivalent to the extension of time-ordered distributions to coincident points. This is achieved by a modified Taylor subtraction on the corresponding test functions. I show that the pullback of this operation to the distributions yields expressions known from differential renormalization. The subtraction is equivalent to BPHZ subtraction in momentum space. Some examples from Euclidean scalar field theory in flat and curved spacetime will be presented.

1. Introduction

Calculations in perturbative QFT are performed primarily in momentum space. The computation of a given contribution to the *S*-matrix is performed by writing down Feynman rules and applying a certain choice of renormalization scheme to the resulting expression. For a reasonable renormalization scheme it should be proved to work for all orders and so produce a finite *S*-matrix as, for example, in the case of BPHZ renormalization.

But today we consider the principle of locality to be of special importance, and hence a local formulation of perturbation theory should exist. This has been elaborated upon by Epstein and Glaser [13] following earlier ideas of Bogoliubov [2]. Their approach is called causal perturbation theory (CPT). Based on a set of axioms they constructed the *S*-matrix as a formal power series inductively. The process of renormalization occurs only once in every step. All lower-order contributions are already renormalized. This corresponds to the determination of all divergent subgraphs in the traditional approach and simplifies the proof of the construction to all orders. The main concept on which CPT is based is its formulation completely in configuration space. In the 1970s there were few applications of this, apart from [1, 8]. This may be due to the fact that Epstein and Glaser used rigorous functional analysis, in which renormalization is defined by an appropriate subtraction on test functions, whereas physicists are used to working with distributions in an integral kernel representation.

Later, Scharf *et al* applied CPT to QED (see [24] and references therein) and to non-Abelian gauge theories [9–12]. But their perturbative calculations were still performed in momentum space.

Alternatively, there is a renormalization scheme called differential renormalization [16, 25]. This works in configuration space and there is no need for a regularization procedure.

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Differential renormalization has been shown to work to all orders by using Bogoliubov's recursion formula [21].

In this paper (following [23]) I show that the integral kernel representation of the Epstein–Glaser subtraction exactly yields differential renormalization. It leads to a formula for the computation of diagrams if all lower order contributions are known, as assumed in the causal approach. Since the fomula turns out to become quite simple in the Euclidean case, I apply it to lowest order examples from Euclidean scalar field theory and use the results for renormalization group computations. In passing to higher orders one had to use a Euclidean version of the causality axiom of CPT [27]. But this is beyond the scope of this paper.

CPT mainly relies on the principles of causality, translation invariance and the singularity structure of the Feynman propagator. Brunetti and Fredenhagen [3] implemented CPT on a globally hyperbolic spacetime by giving a local generalization of translation invariance. Here the local causality structure is preserved and the Feynman propagator is known to have Hadamard form. I show how the corresponding distributions can be renormalized in the Euclidean case. This is achieved by an appropriate translation of their representations from flat spacetime to curved spacetime.

2. The extension of distributions

Following [15, 27] it turns out that renormalization in CPT actually is an extension of distributions from the subspace of test functions whose support does not contain the origin to the space of all test functions. To treat the most general solution of that problem we are concerned with the space of distributions $\mathcal{D}'(\mathbb{R}^n)$, the dual of $\mathcal{D}(\mathbb{R}^n)$, the space of test functions with compact support. Let $\alpha = \{\alpha_1, \ldots, \alpha_n\} \in \mathbb{N}^n$ be a multi-index; we set $|\alpha| = \sum_{i=1}^n \alpha_i$ and $\alpha! = \prod_{i=1}^n \alpha_i!$ and

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} \ldots \partial {x_n}^{\alpha_n}} \tag{1}$$

is a partial differential operator of order $|\alpha|$.

Remark. *Note that all operations on distributions like differentiation and transformations of their arguments are defined by the corresponding operations on test functions.*

This fact is referred to as *'in the sense of distributions'*. Writing

$$T(\varphi) = \int \mathrm{d}^n x\, T(x)\varphi(x) \qquad T \in \mathcal{D}'(\mathbb{R}^n) \qquad \varphi \in \mathcal{D}(\mathbb{R}^n) \tag{2}$$

we call $T(x)$ the integral kernel of T. Let $\mathcal{D}(\mathbb{R}^n \setminus \{0\}) = \{\varphi \in \mathcal{D}(\mathbb{R}^n) | 0 \notin \mathrm{supp}(\varphi)\}$ denote the subspace of test functions whose support does not contain the origin and $\mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ its dual†. Now we state the following problem.

Problem. *Given a distribution* ${}^0T \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$, *how can we construct an extension* $T \in \mathcal{D}'(\mathbb{R}^n)$, *such that* ${}^0T(\varphi) = T(\varphi)$ *for* $\varphi \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$*?*

The solution of this problem requires the introduction of a quantity that measures the singularity of the distribution at the origin [26].

Definition 1. *A distribution* $T \in \mathcal{D}'(\mathbb{R}^n)$ *has scaling degree* s *at* $x = 0$, *if*

$$s = \inf\{s' \in \mathbb{R} | \lambda^{s'} T(\lambda x) \xrightarrow{\lambda \to 0} 0 \textit{ in the sense of distributions}\}. \tag{3}$$

† The existence of the extension is guaranteed by the Hahn–Banach theorem. A solution for homogenous distributions can be found in [19, ch III.2].

Let s be denoted by $\operatorname{scal deg}(T)$, *and define* $\operatorname{sing ord}(T) := [s] - n$, *the singular order*‡.

The definition also holds if $T \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$. We take the δ-distribution as an example

Example 1. *For $\delta \in \mathcal{D}'(\mathbb{R}^n)$ one has: $\delta(\lambda x) = |\lambda|^{-n}\delta(x)$. The scaling degree of δ is n, the singular order is zero.*

The scaling degree of some special compositions of distributions can be computed quite easily. We state the following proposition.

Proposition 1. *Let $T \in \mathcal{D}'(\mathbb{R}^n)$ or $\mathcal{D}'(\mathbb{R}^n \setminus \{0\})$,* $\operatorname{scal deg}(T) = s$ *and β be a multi-index.*

(I) $\quad \operatorname{scal deg}(x^\beta T) = s - |\beta|$.

(II) $\quad \operatorname{scal deg}(\partial^\beta T) = s + |\beta|$.

(III) $\quad \operatorname{scal deg}(w) \leqslant 0 \qquad \operatorname{scal deg}(wT) \leqslant s \quad w \in \mathcal{D}(\mathbb{R}^n)$.

(IV) $\quad \operatorname{scal deg}(T_1 \otimes T_2) = s_1 + s_2 \qquad$ if $\quad \operatorname{scal deg}(T_i) = s_i \quad i = 1, 2$.

The proof is skipped; we only note that all statements follow directly from the translation of the words ‘in the sense of distributions’ and the use of the Banach–Steinhaus theorem (principle of uniform boundedness, applied to distributions) on point (*III*).

Example 2. *The scaling degree of $\delta^{(\alpha)} \in \mathcal{D}'(\mathbb{R}^n)$ is $|\alpha| + n$. The singular order is $|\alpha|$.*

The solution of the problem depends on the sign of the singular order. Let us consider the simple case first.

Theorem 2. *Let ${}^0T \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ with scaling degree $s < n$. Then there exists a unique $T \in \mathcal{D}'(\mathbb{R}^n)$ with scaling degree s and $T(\varphi) = {}^0T(\varphi)$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$.*

The proof can be found in [3]. If the scaling degree is not smaller than the space dimension, the singular order ω is zero or positive. In that case theorem 2 guarantees a unique extension on test functions that vanish at the origin up to order ω. Thus a general extension can be defined after performing a projection into that subspace. This is achieved by a kind of modified Taylor subtraction, called the W-operation.

W-operation. Let $\mathcal{D}^\omega(\mathbb{R}^n)$ be the subspace of test functions vanishing up to order ω at 0. Define

$$\begin{aligned} &W_{(\omega;w)} : \mathcal{D}(\mathbb{R}^n) \mapsto \mathcal{D}^\omega(\mathbb{R}^n) \qquad \varphi \mapsto W_{(\omega;w)}\varphi \\ &(W_{(\omega;w)}\varphi)(x) = \varphi(x) - w(x) \sum_{|\alpha| \leqslant \omega} \frac{x^\alpha}{\alpha!} \left(\partial^\alpha \frac{\varphi}{w}\right)(0) \end{aligned} \tag{4}$$

with $\quad w \in \mathcal{D}(\mathbb{R}^n) \quad w(0) \neq 0$.

The action of $W_{(\omega;w)}$ on φ can be written as

$$(W_{(\omega;w)}\varphi)(x) = \sum_{|\beta| = \omega + 1} x^\beta \varphi_\beta(x) \tag{5}$$

with $\varphi_\beta \in \mathcal{D}(\mathbb{R}^n)$. It has the nice property

$$W_{(\omega;w)} w\varphi = w W_{(\omega;1)}\varphi. \tag{6}$$

With $(\partial^\alpha x^\gamma)(0) = \gamma!\delta^\gamma_\alpha$ it follows for $|\gamma| \leqslant \omega$:

$$W_{(\omega;w)} w x^\gamma = w W_{(\omega;1)} x^\gamma \equiv 0. \tag{7}$$

Now we can discuss the general case.

‡ $[s]$ is the largest integer that is smaller than or equal to s.

Theorem 3. *Let $^0T \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ with scaling degree $s \geqslant n$. Given $w \in \mathcal{D}(\mathbb{R}^n)$ with $w(0) \neq 0$, a multi-index α, $|\alpha| \leqslant \omega$ and constants $C^\alpha \in \mathbb{C}$, then there is exactly one distribution $T' \in \mathcal{D}'(\mathbb{R}^n)$ with scaling degree s and the following properties:*

(I) $\langle T', \varphi \rangle = \langle {}^0T, \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$
(II) $\langle T', wx^\alpha \rangle = C^\alpha$.

T' is given by:

$$\langle T', \varphi \rangle = \langle T, W_{(\omega;w)}\varphi \rangle + \sum_{|\alpha| \leqslant \omega} \frac{C^\alpha}{\alpha!} \left(\partial^\alpha \frac{\varphi}{w} \right)(0). \tag{8}$$

Here T is the unique extension by theorem 2, $W_{(\omega;w)}$ is given by (4) and ω is the singular order of ^0T.

The proof can be found in [3]. We see that in the case of non-negative singular order, the extension is not unique. It is fixed by a finite set of complex numbers C^α. Let us look at the next example.

Example 3. *The nth power of the scalar Feynman propagator $(\mathrm{i}\Delta_F)^n(x) = \theta(x^0){\Delta_+}^n(x) + \theta(-x^0){\Delta_+}^n(-x)$ is a distribution on $\mathbb{R}^4 \setminus \{0\}$. We compute the scaling degree of ${\Delta_+}^n$.*

$$\begin{aligned}
{\Delta_+}^n(\lambda x) &= (2\pi)^{-3n} \int \prod_{i=1}^n \frac{\mathrm{d}^3\boldsymbol{p}_i}{2\omega_{\boldsymbol{p}_i}} \mathrm{e}^{\sum_{i=1}^n (-\mathrm{i}\omega_{\boldsymbol{p}_i}\lambda x^0 + \mathrm{i}\boldsymbol{p}_i\lambda \boldsymbol{x})} \\
&= \lambda^{-2n}(2\pi)^{-3n} \int \prod_{i=1}^n \frac{\mathrm{d}^3\boldsymbol{p}_i}{2\sqrt{(\lambda m)^2 + {\boldsymbol{p}_i}^2}} \mathrm{e}^{\sum_{i=1}^n (-\mathrm{i}\sqrt{(\lambda m)^2 + {\boldsymbol{p}_i}^2}x^0 + \mathrm{i}\boldsymbol{p}_i\boldsymbol{x})} \\
&= \lambda^{-2n} {\Delta_+}^n(x, \lambda m) \\
&\xrightarrow{\lambda \to 0} \lambda^{-2n} {D_+}^n(x).
\end{aligned}$$

Here D_+ denotes the massless scalar two-point function. Hence the scaling degree is $2n$. The application of the W-operation with $\omega = 2n - 4$ yields the extension to all test functions. The computation can be done similarly for the Euclidean propagator.

We turn to another example that seems to have caused some confusion in classical physics (see e.g. [14]).

Example 4 (The self energy of the electron). *In electrostatics the electric potential of an electron at the origin is given by the Green function of the Laplace equation in three dimensions.*

$$\Delta\phi = -4\pi\rho = 4\pi e\delta \qquad \Rightarrow \qquad \phi = -\frac{e}{r} \in \mathcal{D}'(\mathbb{R}^3).$$

The electric field is

$$\boldsymbol{E} = -\nabla\phi = -\frac{e\boldsymbol{r}}{r^3} \in \mathcal{D}'(\mathbb{R}^3).$$

Since $\operatorname{sing\,supp}(\boldsymbol{E}) = \{0\}$ *it follows:*

$$\boldsymbol{E}^2 = \frac{e^2}{r^4} \in \mathcal{D}'(\mathbb{R}^3 \setminus \{0\}).$$

The singular order is one. Hence there is an extension to all test functions by the W-operation. We can define the energy density $U = \boldsymbol{E}^2$ as the following distribution:

$$\langle U, \varphi \rangle := \langle \boldsymbol{E}^2, W_{(1;w)}\varphi \rangle. \tag{9}$$

An electron at rest has self-energy $E = 1/(4\pi)\,\langle U, 1\rangle$. The choice of $\varphi \equiv 1$ is possible due to sufficient convergence at long range. The same holds for w in (9).

$$E = \frac{1}{4\pi}\left\langle U, W_{(1;1)}1\right\rangle + C^0 = C^0$$

as all C^α, $|\alpha| = 1$ vanish. We can determine C^0 by the requirement that the mass of the electron is purely electromagnetic, i.e.

$$E = mc^2.$$

In the following I will suppress the distinction between ${}^0T$ and T in the case that the scaling degree is smaller than the space dimension. This should lead to no confusion since the extension is unique in that case.

2.1. *The integral kernel representation*

Using standard vocabulary we call the extended distribution in theorem 3 a *renormalization*. Next we will work out its integral kernel. If we set all C^α to zero we have the following definition.

Definition 2. *Let $T \in \mathcal{D}^{\omega'}(\mathbb{R}^n)$ with $\operatorname{sing\,ord}(T) = \omega$. The integral kernel $T_{R(\omega;w)}$ of its extension is given by*

$$\left\langle T_{R(\omega;w)}, \varphi\right\rangle := \left\langle T, W_{(\omega;w)}\varphi\right\rangle. \tag{10}$$

Furthermore, we consider a family of distributions T_t that depend continuously on a real parameter t. If K is a real compact interval then $\int_K \mathrm{d}t\,\langle T_t, \varphi\rangle$ exists as a Riemannian integral. We define

$$\left\langle \int_K \mathrm{d}t\, T_t, \varphi\right\rangle := \int_K \mathrm{d}t\,\langle T_t, \varphi\rangle \tag{11}$$

in the sense of distributions. Now we have the following proposition.

Proposition 4. *The integral kernel $(T_{R(\omega;w)}w)$ is given by:*

$$(T_{R(\omega;w)}w)(x) = (-)^{\omega+1}(\omega+1)\sum_{|\beta|=\omega+1}\partial^\beta\frac{x^\beta}{\beta!}\int_0^1 \mathrm{d}t\,\frac{(1-t)^\omega}{t^{n+\omega+1}}T\left(\frac{x}{t}\right)w\left(\frac{x}{t}\right). \tag{12}$$

Proof. The Taylor expansion of φ at the origin is:

$$\varphi(x) = \sum_{|\alpha|=0}^{\omega}\frac{x^\alpha}{\alpha!}(\partial^\alpha\varphi)(0) + (\omega+1)\sum_{|\beta|=\omega+1}\frac{x^\beta}{\beta!}\int_0^1 \mathrm{d}t\,(1-t)^\omega(\partial^\beta\varphi)(tx). \tag{13}$$

Hence $(W_{(\omega;1)}\varphi)(x)$ is the Taylor rest term of order $\omega+1$. Writing (12) as $(T_{R(\omega;w)}w) = \int_0^1 \mathrm{d}t\,(T_{R(\omega;w)}w)_t$ we find using (11):

$$\begin{aligned}
\left\langle \int_0^1 \mathrm{d}t\,(T_{R(\omega;w)}w)_t, \varphi\right\rangle &= (\omega+1)\int_0^1 \mathrm{d}t\,(1-t)^\omega\int \mathrm{d}^n x\sum_{|\beta|=\omega+1}\frac{x^\beta}{\beta!}T(x)w(x)(\partial^\beta\varphi)(tx)\\
&= \left\langle T, wW_{(\omega;1)}\varphi\right\rangle\\
&= \left\langle T, W_{(\omega;w)}w\varphi\right\rangle\\
&= \left\langle T_{R(\omega;w)}, w\varphi\right\rangle\\
&= \left\langle T_{R(\omega;w)}w, \varphi\right\rangle
\end{aligned}$$

where we used equation (6). □

Note that the differential operator in (12) is a *weak* derivative. If w had no zeros ($\Rightarrow w \notin \mathcal{D}(\mathbb{R}^n)$), the integral kernel would result from a simple division. Later we encounter a well known example of this.

To achieve a similar representation for the whole distribution we put a restriction on the test function w in (4). Let $w(0) = 1$ and $(\partial^\alpha w)(0) = 0$, for $|\alpha| \leqslant \omega$.

Remark. *This is no real loss of generality since for a given* $v \in \mathcal{D}(\mathbb{R}^n)$, $v(0) \neq 0$, *the projection*

$$v \mapsto w = v \sum_{|\alpha| \leqslant \omega} \frac{x^\alpha}{\alpha!} \left(\partial^\alpha \frac{1}{v} \right)(0) \in \mathcal{D}(\mathbb{R}^n) \tag{14}$$

yields $\quad (\partial^\gamma w)(0) = \delta_0^\gamma \qquad$ *for* $\quad |\gamma| \leqslant \omega.$

Then $(1 - w)$ vanishes up to order ω at 0:

$$W_{(\omega;w)}(1 - w)\varphi = (1 - w)\varphi \tag{15}$$

$$\left\langle (1 - w)T_{R(\omega;w)}, \varphi \right\rangle = \langle (1 - w)T, \varphi \rangle\,. \tag{16}$$

Now the integral kernel is given by the following lemma.

Lemma 5. *With the above restrictions on* $w \in \mathcal{D}(\mathbb{R}^n)$, *the renormalized distribution* $T_{R(\omega;w)}$ *has the following integral kernel:*

$$\begin{aligned} T_{R(\omega;w)}(x) = (-)^\omega(\omega + 1) \sum_{|\beta| = \omega+1} \partial^\beta \frac{x^\beta}{\beta!} \Bigg[&- \int_0^1 \mathrm{d}t \, \frac{(1 - t)^\omega}{t^{n+\omega+1}} T\left(\frac{x}{t}\right) w\left(\frac{x}{t}\right) \\ &+ \int_1^\infty \mathrm{d}t \, \frac{(1 - t)^\omega}{t^{n+\omega+1}} T\left(\frac{x}{t}\right) (1 - w)\left(\frac{x}{t}\right) \Bigg]. \end{aligned} \tag{17}$$

Proof. The first term of of (17) is the integral kernel of $wT_{R(\omega;w)}$. A simple computation shows that the second term smeared with φ yields $\langle T, (1 - w)\varphi \rangle$. Equation (16) completes the proof. $\square$

If $n = 4m$, we can write

$$k! \sum_{|\beta| = k} \partial^\beta \frac{x^\beta}{\beta!} = \natural \left(\sum_{|\beta| = 1} \partial^\beta x^\beta \right)^k \natural = \natural (\partial_{\mu_1} x^{\mu_1} + \cdots + \partial_{\mu_m} x^{\mu_m})^k \natural \tag{18}$$

$$\text{with} \quad x = \begin{pmatrix} x^{\mu_1} \\ \vdots \\ x^{\mu_m} \end{pmatrix} \in \mathbb{R}^{4m}.$$

Here $\natural \ldots \natural$ denotes the ordering of differential operators to the left of the coordinates.

To perform some computations with formula (17) it would be desirable to abandon the requirement of w being a test function. Let $w_m \in \mathcal{D}(\mathbb{R}^n)$ be a sequence with $\lim_{m \to \infty} w_m =: w \in \mathcal{D}'(\mathbb{R}^n)$. If $\lim_{m \to \infty} \left\langle T_{R(\omega;w_m)}, \varphi \right\rangle \in \mathbb{C}$ exists $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$, we will allow w to be used in the renormalization procedure. Let us consider $T \in \mathcal{D}^{\omega\prime}(\mathbb{R}^n)$ with singular order ω and $\operatorname{sing\,supp}(T) = \{0\}$. Choose $w(x) = \theta(1/M - |x|) =: \theta^<(x)$, $M \in \mathbb{R}$, $M > 0$, where $|\cdot|$ denotes the Euclidean norm. Since $\operatorname{sing\,supp}(T) \cap \operatorname{sing\,supp}(\theta^<) = \emptyset$ and $\theta^<$ has compact support, the pointwise product $\theta^< T \in \mathcal{E}^{\omega\prime}(\mathbb{R}^n) \subset \mathcal{D}^{\omega\prime}(\mathbb{R}^n)$ exists†.

† $\mathcal{E} = \mathcal{C}^\infty$ and $\mathcal{E}'$ is the space of distributions with compact support.

Applying (17) yields

$$T_{R(\omega;\theta^<)}(x) = (-)^{\omega}(\omega+1)\sum_{|\beta|=\omega+1}\partial^{\beta}\frac{x^{\beta}}{\beta!}\int_1^{M|x|}\mathrm{d}t\,\frac{(1-t)^{\omega}}{t^{n+\omega+1}}T\left(\frac{x}{t}\right)$$
$$=: T_R^M(x). \tag{19}$$

For an arbitrary choice of w, a scale M has to be introduced for dimensional reasons. This allows writing $w(Mx)$ as a function of a dimensionless argument. The dependence of the counterterms on the scale can easily be computed:

$$M\frac{\partial}{\partial M}\left\langle T_{R(\omega;w(Mx))},\varphi\right\rangle = \sum_{|\alpha|\leqslant\omega}B^{\alpha}\left\langle\delta^{(\alpha)},\varphi\right\rangle \tag{20}$$

$$\text{with}\quad B^{\alpha} = \frac{(-)^{|\alpha|+1}}{\alpha!}\left\langle T, Mx^{\mu}(\partial_{\mu}w)(Mx)x^{\alpha}\right\rangle. \tag{21}$$

For $w=\theta^<$ we get

$$B^{\alpha} = \frac{(-)^{|\alpha|}}{\alpha!}\left\langle T,\delta\left(\frac{1}{M}-|x|\right)x^{\alpha}\right\rangle. \tag{22}$$

2.2. Momentum space and BPHZ renormalization

Since the Fourier transformation is a map $\mathcal{D}\mapsto\mathcal{S}$ we have to restrict our distributional space to $\mathcal{S}'\subset\mathcal{D}'$‡.

We remind the reader that the Fourier transformation is defined in the sense of distributions, i.e. $\hat{T}(\varphi) := T(\hat{\varphi})$.

First we start with the definition of the moments of a test function:

$$K^{\alpha}(\psi) := \int \mathrm{d}^n x\, x^{\alpha}\psi(x) \qquad \psi\in\mathcal{S}(\mathbb{R}^n). \tag{23}$$

Let $\mathcal{S}_{\omega}(\mathbb{R}^n) := \{\psi\in\mathcal{S}(\mathbb{R}^n), K^{\alpha}(\psi)=0, |\alpha|\leqslant\omega\}$ be the subspace of test function with vanishing moments up to order ω, then it follows: $\psi\in\mathcal{S}^{\omega}\Rightarrow\check{\psi}\in\mathcal{S}_{\omega}$. Choosing w as in lemma 5 we have:

$$K^{\alpha}(\hat{w}) = 0 \qquad \text{for}\quad 0<|\alpha|\leqslant\omega \qquad K^0(\hat{w}) = (2\pi)^n \tag{24}$$

and by a simple computation:

$$K^{\gamma}(\partial^{\alpha}\widehat{w}) = (-)^{|\gamma|}\gamma!\delta_{\alpha}^{\gamma}(2\pi)^n \qquad \gamma\leqslant\alpha. \tag{25}$$

The Fourier transformation of $W\varphi$ is:

$$(W_{(\omega;w)}\varphi)^{\vee}(p) = \check{\varphi}(p) - \frac{1}{(2\pi)^n}\sum_{|\alpha|\leqslant\omega}\frac{(-)^{|\alpha|}}{\alpha!}(\widehat{wx^{\alpha}})(-p)\left\langle\delta^{(\alpha)},\varphi\right\rangle \tag{26}$$

$$= \check{\varphi}(p) - \frac{1}{(2\pi)^n}\sum_{|\alpha|\leqslant\omega}\frac{K^{\alpha}(\check{\varphi})}{\alpha!}(\partial^{\alpha}\hat{w})(-p). \tag{27}$$

Using (25) we get:

$$K^{\gamma}((W_{(\omega;w)}\varphi)^{\vee}) = 0 \qquad |\gamma|\leqslant\omega \tag{28}$$

‡ Let $\mathcal{S}$ be the space of $\mathcal{C}^{\infty}$ functions of rapid decrease and $\mathcal{S}'$ its dual. We use the convention

$$\hat{\psi}(p) = \int \mathrm{d}^n x\,\psi(x)\mathrm{e}^{\mathrm{i}px}.$$

so $W_{(\omega;w)}{}^{\vee}$ is actually a projector $\mathcal{S} \mapsto \mathcal{S}_\omega$. With

$$\left\langle T_{R(\omega;w)}, \varphi \right\rangle = \left\langle \widehat{T_{R(\omega;w)}}, \check{\varphi} \right\rangle = \left\langle \hat{T}, (W_{(\omega;w)}\varphi)^{\vee} \right\rangle \tag{29}$$

the integral kernel is given by

$$\widehat{T_{R(\omega;w)}}(k) = \hat{T}(k) - \sum_{|\alpha|=0}^{\omega} \frac{k^\alpha}{\alpha!} (\partial^\alpha \widehat{Tw})(0). \tag{30}$$

This subtraction should be understood in the sense of distributions, i.e. the subtraction on the test functions has to occur before smearing out. It looks similar to BPHZ renormalization which is a Taylor subtraction at arbitrary momentum q. We shall compute the corresponding w. Let

$$\hat{T}_R^q(k) := \hat{T}(k) - \sum_{|\alpha|=0}^{\omega} \frac{(k-q)^\alpha}{\alpha!} (\partial^\alpha \hat{T})(q) \tag{31}$$

denote the BPHZ renormalized distribution in momentum space. Using the Taylor rest expression similar to (13) and performing Fourier transformation we get:

$$T_R^q(x) = (-)^{\omega+1}(\omega+1)\mathrm{e}^{-\mathrm{i}qx} \sum_{|\beta|=\omega+1} \partial^\beta \frac{x^\beta}{\beta!} \int_0^1 \mathrm{d}t \, \frac{(1-t)^\omega}{t^{n+\omega+1}} T\left(\frac{x}{t}\right) \mathrm{e}^{\mathrm{i}\frac{qx}{t}} \tag{32}$$

and comparing with (12)

$$= T_{R(\omega;\mathrm{e}^{\mathrm{i}qx})}. \tag{33}$$

Here, the equivalence of the subtraction procedure in Epstein–Glaser and BPHZ renormalization can be seen explicitly.

3. Causal perturbation theory

In the following I give a very brief summary of CPT. A complete description can be found in [13, 24].

We start with the S-Matrix as a formal power series:

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n \; T_n(x_1, \ldots, x_n) g(x_1) \ldots g(x_n). \tag{34}$$

It is an operator valued functional. The function $g \in \mathcal{D}(\mathbb{R}^4)$ plays the role of a coupling 'constant'. The T_n are operator-valued distributions in Fock space. They are called time-ordered functions and involve free fields only. Epstein and Glaser stated six axioms from which the time-ordered functions can be computed recursively. Here we cite only the most important one called the causality axiom:

$$T_n(x_1, \ldots, x_n) = T_k(x_1, \ldots, x_k) T_{n-k}(x_{k+1}, \ldots, x_n) \tag{35}$$

if all points $x_{k+1}, \ldots, x_n$ are not in the causal past of $x_1, \ldots, x_k$. The inductive construction starts with $T_1 = \mathcal{L}_{\mathrm{int}}$. We assume that $T_{n'}(x_1, \ldots, x_{n'})$ for all $n' < n$ exist as a sum of products of a symmetric translation invariant numerical distribution and a Wick polynomial of fields. The scaling degree of the numerical distribution is known at coincident points. Now T_n can be constructed up to the total diagonal $x_1 = \ldots = x_n$ by the causality axiom. Wick's theorem ensures the required form. The numerical distributions have an extension to the diagonal which is the origin in $\mathbb{R}^{4n-4}$ because of translation invariance. The Wick polynomials are already defined as operator-valued distributions on the whole space. Let me emphasize that the translation invariance of the numerical distributions plays a crucial role in the whole construction.

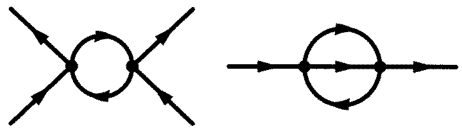

Figure 1. Feynman graphs corresponding to (40), (41).

4. Applications

I now give some examples from ϕ^4-theory in lowest order. The Lagrangian for the self-interacting scalar field is

$$\mathcal{L}(x) = \frac{1}{2} :\partial_\mu \phi(x) \partial^\mu \phi(x): - \frac{m^2}{2} :\phi^2(x): - \frac{\lambda}{4!} :\phi^4(x): . \tag{36}$$

T_1 is given by the interaction term:

$$T_1(x) = -\frac{\lambda}{4!} :\phi^4(x): . \tag{37}$$

Causality (35) implies T_2 for non-coincident points.

$$T_2(x_1, x_2) = \begin{cases} T_1(x_1)T_1(x_2) & \text{if } x_1^0 > x_2^0 \\ T_1(x_2)T_1(x_1) & \text{if } x_2^0 > x_1^0 \end{cases}$$

$$= \frac{\lambda^2}{(4!)^2}[:\phi^4(x_1)\phi^4(x_2): \tag{38}$$

$$+\, 16\mathrm{i}\Delta_F(x_1 - x_2) :\phi^3(x_1)\phi^3(x_2): \tag{39}$$

$$+\, 72(\mathrm{i}\Delta_F)^2(x_1 - x_2) :\phi^2(x_1)\phi^2(x_2): \tag{40}$$

$$+\, 96(\mathrm{i}\Delta_F)^3(x_1 - x_2) :\phi(x_1)\phi(x_2): \tag{41}$$

$$+\, 24(\mathrm{i}\Delta_F)^4(x_1 - x_2)]. \tag{42}$$

To give some explicit results for the numerical distributions we now turn to their corresponding Euclidean counterparts. The singular support of these distributions is the origin only. Hence we can use (19) for the renormalization.

4.1. *The massless theory*

The Green function of the Laplace equation in four dimensions is

$$D_F(x) = \frac{1}{4\pi^2}\frac{1}{x^2}. \tag{43}$$

It has singular order -2. Consider the contribution to the two-point, respectively, four-point vertex function.

4.1.1. The one-loop graph. The 'fish' graph (40) has singular order zero, so we get

$$D_F{}^2|_R^M(x) = \frac{1}{(2\pi)^4}\frac{1}{2}\partial_\mu x^\mu \frac{\ln(M^2x^2)}{(x^2)^2}. \tag{44}$$

Since all extensions only differ by a δ-term we have

$$M\frac{\partial}{\partial M} D_F{}^2|_R^M(x) = \frac{1}{8\pi^2}\delta(x) \qquad \text{or} \qquad \partial_\mu \frac{x^\mu}{(x^2)^2} = 2\pi^2\delta(x) \tag{45}$$

by (22). This also could have been seen by expressing $\frac{x^\mu}{(x^2)^2} = -\frac{1}{2}\partial^\mu \frac{1}{x^2}$, which is unique by theorem 2.

4.1.2. The two-loop graph. The singular order of the 'setting sun' (41) is two. Hence we have

$$D_F{}^3|_R^M(x) = \frac{1}{(2\pi)^6}\frac{1}{4}\partial_\mu\partial_\nu\partial_\sigma x^\mu x^\nu x^\sigma \frac{1}{(x^2)^3}[\ln(M^2x^2) + M^2x^2 - 4M\sqrt{x^2} + 3]. \tag{46}$$

Terms two, three and four have singular order 0, 1 and 2. As they are zero outside the origin, they must be proportional to δ and its first, respectively second, derivatives. As there is no Euclidean invariant combination of ∂ and δ, term two has to be zero. Using (22) we get by comparison with $M\frac{\partial}{\partial M}$ on (46):

$$\partial_\mu\partial_\nu\partial_\sigma x^\mu x^\nu x^\sigma \frac{1}{(x^2)^3} = \frac{\pi^2}{2}\Delta\delta(x) \tag{47}$$

$$\partial_\mu\partial_\nu\partial_\sigma x^\mu x^\nu x^\sigma \frac{1}{(x^2)^2} = 4\pi^2\delta(x) \tag{48}$$

$$\partial_\mu\partial_\nu\partial_\sigma x^\mu x^\nu x^\sigma \frac{1}{\sqrt{x^2}^5} \equiv 0 \tag{49}$$

where Δ is the Laplacian. Now we turn to the massive theory.

4.2. The massive theory

The Green function of the Euclidean Klein–Gordon equation is

$$\Delta_F(x) = \frac{1}{(2\pi)^2}\frac{mK_1(m\sqrt{x^2})}{\sqrt{x^2}}. \tag{50}$$

As $K_1(x) \propto 1/x$ for $x \to 0$, the singular order of $\Delta_F = -2$. We compute as follows.

4.2.1. The one-loop graph. With $\operatorname{sing\,ord}(\Delta_F{}^2) = 0$ and (19) we get:

$$(\Delta_F{}^2)\|_R^M(x) = \frac{1}{32\pi^4}\partial_\mu x^\mu \left\{\frac{m^2}{x^2}[K_1{}^2(m\sqrt{x^2}) - K_0(m\sqrt{x^2})K_2(m\sqrt{x^2})] \right.$$
$$\left. -\frac{m^2}{M^2}\frac{1}{(x^2)^2}\left[K_1{}^2\left(\frac{m}{M}\right) - K_0\left(\frac{m}{M}\right)K_2\left(\frac{m}{M}\right)\right]\right\}. \tag{51}$$

By writing $x^\mu \ldots = \partial^\mu \ldots$ which is unique we can express it as:

$$= \frac{m^2}{2(2\pi)^4}\Delta\left\{K_0{}^2(m\sqrt{x^2}) - K_1{}^2(m\sqrt{x^2}) + \frac{K_0(m\sqrt{x^2})K_1(m\sqrt{x^2})}{m\sqrt{x^2}}\right.$$
$$\left. +\frac{1}{2M^2x^2}\left[K_1{}^2\left(\frac{m}{M}\right) - K_0\left(\frac{m}{M}\right)K_2\left(\frac{m}{M}\right)\right]\right\}. \tag{52}$$

This can be compared to the corresponding expression in [18].

4.2.2. The two-loop graph. Applying formula (19) leads to an integral $\int \mathrm{d}s\,(s-\text{const})^2\mathrm{K}_1{}^3(\mathrm{s})$ that is difficult to solve. But if we use the expansion

$$\frac{m^3K_1{}^3(mr)}{r^3} = \frac{1}{r^6} + \frac{3m^2}{4r^4}\left(\ln\left(\frac{m^2r^2}{4}\right) + \ln(\gamma^2) - 1\right) + R(r^{-2}) \tag{53}$$

we can renormalize the first and second summand with singular order 2, 0 respectively. With

$$\left.\frac{\ln\left(\frac{m^2x^2}{4}\right)}{(x^2)^2}\right|_R^M = \frac{1}{4}\partial_\mu x^\mu \frac{\ln\left(\frac{m^4x^2}{16M^2}\right)\ln(M^2x^2)}{(x^2)^2} \tag{54}$$

and the previous results from section 3.1 we get:

$$\begin{aligned}(\Delta_F{}^3)\big|^{M}_{MR}(x) = \frac{1}{(2\pi)^6}\Bigg\{&\frac{1}{4}\partial_\mu\partial_\nu\partial_\sigma x^\mu x^\nu x^\sigma \frac{1}{(x^2)^3}[\ln(M^2x^2) + M^2x^2 + 3] \\ &+\frac{3m^2}{16}\partial_\mu x^\mu \frac{1}{(x^2)^2}\left[\ln\left(\frac{m^4x^2}{16M^2}\right) + 2\ln(\gamma^2) - 2\right]\ln(M^2x^2) + R(x^{-2})\Bigg\}\end{aligned} \tag{55}$$

where the subscript MR denotes the 'minimal renormalization', i.e. every summand is renormalized with its singular order.

4.3. *The renormalization group*

The parameter M plays the role of a renormalization scale. It enters the theory by purely dimensional reasons as an argument of the 'function' w. We give the lowest-order contributions to the β, γ and γ_m functions in the renormalization group. They can be read off by solving the renormalization group equation

$$\left[M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g} + m\gamma_m(g)\frac{\partial}{\partial m} - n\gamma(g)\right]\Gamma_n(x_1,\ldots,x_n) = 0 \tag{56}$$

to order g^2, where $g = \lambda/(16\pi^2)$. Using the expansions

$$\beta(g,m,M) = M\frac{\partial g}{\partial M} = \sum_{n=2}^{\infty}\beta_n g^n \tag{57}$$

$$\gamma_m(g,m,M) = M\frac{\partial \ln m}{\partial M} = \sum_{n=2}^{\infty}\gamma_{m,n} g^n \tag{58}$$

$$\gamma(g,m,M) = \frac{M}{2}\frac{\partial \ln Z}{\partial M} = \sum_{n=2}^{\infty}\gamma_n g^n \tag{59}$$

we find $\beta_2 = 3$ and $\gamma_2 = \frac{1}{12}$ for the massless theory. Here we had to add a term to the setting sun proportional to (48) to achieve $\gamma_m \equiv 0$.

In the massive theory we find

$$\beta_2 = 3\frac{m^2}{M^2}K_1{}^2\left(\frac{m}{M}\right) \tag{60}$$

hence $\beta_2 \to 3$ if $M \to \infty$. This result also holds for the corresponding minimal renormalization. In that scheme we get

$$\gamma_2 = \tfrac{1}{12} \tag{61}$$

$$\gamma_{m,2} = \frac{2M^2}{3m^2} + \frac{1}{2}\ln\left(\frac{m^2}{4M^2}\right) + \ln(\gamma) - \frac{5}{12} \tag{62}$$

by using (55).

4.4. *Curved spacetime*

Let $\mathcal{M}$ be a globally hyperbolic manifold with a metric g. Wick polynomials were defined in [4] using techniques from microlocal analysis. Then CPT was implemented in [3] for scalar ϕ^4-theory. The Feynman propagator is known to have Hadamard structure [20]:

$$\Delta_F \propto \frac{\Delta^{\frac{1}{2}}}{2\sigma} + v\ln(2\sigma) + w \tag{63}$$

where $\sigma,\ \Delta,\ v$, and w are smooth functions on the manifold and an appropriate $\mathrm{i}\epsilon$ regularization has to be chosen. By using a chart it can be seen to have the same scaling degree as in flat spacetime. The function σ is half the square of the geodesic distance which is unique in every sufficiently small neighbourhood. Let $g = \det(g_{ab})$. The Van–Vleck–Morette determinant

$$\Delta(p, p') = -\frac{1}{\sqrt{g(p)\, g(p')}} \det(-\sigma_{ab'}(p, p')) \qquad p, p' \in \mathcal{M} \tag{64}$$

fulfils the following differential equation:

$$\nabla_a(\Delta\,\sigma^a) = 4\Delta. \tag{65}$$

The vector index on σ denotes the covariant derivative as usual. If we expand

$$v = \sum_{n=0}^{\infty} v_n \sigma^n \qquad w = \sum_{n=0}^{\infty} w_n \sigma^n \tag{66}$$

in powers of σ, the coefficients (except w_0) are determined by the Hadamard recursion relations, see [7].

Following [3] the causal construction of the second-order S-matrix is the same as in flat spacetime. The only step left is to perform the extension to coincident points. Therefore we turn to the corresponding Euclidean distributions, so we can transfer our results from the previous examples†.

Differential renormalization in curved spacetime has already been used in [6]. But our results may be compared with other renormalization techniques too [5,22]. Again we only use the minimal renormalization scheme.

4.4.1. The one-loop graph. This is given by

$$\Delta_F{}^2|^M_{MR} = \frac{1}{16\pi^4}\left(\left.\frac{\Delta}{(2\sigma)^2}\right|^M_R + 2\frac{\Delta^{\frac{1}{2}} v \ln(2\sigma)}{2\sigma} + 2\frac{\Delta^{\frac{1}{2}} w}{2\sigma} + 2vw\ln(2\sigma) + v^2\ln^2(2\sigma) + w^2\right). \tag{67}$$

Only the first term needs renormalization and we get

$$\left.\frac{\Delta}{(2\sigma)^2}\right|^M_R = \frac{1}{2}\nabla_a\sigma^a \frac{\Delta\ln(2M^2\sigma)}{(2\sigma)^2}. \tag{68}$$

4.4.2. The two-loop graph. Here we use the expansion (66) to determine the terms that have to be renormalized. Then we are left with

$$\begin{aligned}\Delta_F{}^3|^M_{MR} = \frac{1}{(2\pi)^6}\Bigg(&\left.\frac{\Delta^{\frac{3}{2}}}{(2\sigma)^3}\right|^M_R + 3\left.\frac{\Delta v_0\ln(2\sigma)}{(2\sigma)^2}\right|^M_R + 3\left.\frac{\Delta w_0}{(2\sigma)^2}\right|^M_R + 3\frac{\Delta\bar{v}\ln(2\sigma)}{2\sigma}\\ &+3\frac{\Delta\bar{w}}{2\sigma} + 3\frac{\Delta^{\frac{1}{2}}v^2\ln^2(2\sigma)}{2\sigma} + 6\frac{\Delta^{\frac{1}{2}}vw\ln(2\sigma)}{2\sigma} + 3\frac{\Delta^{\frac{1}{2}}w^2}{2\sigma}\\ &+v^3\ln^3(2\sigma) + 3v^2w\ln^2(2\sigma) + 2vw^2\ln(\sigma) + w^3\Bigg)\end{aligned} \tag{69}$$

with

$$\bar{v} = \frac{1}{2}\sum_{n=0}^{\infty} v_{n+1}\sigma^n \qquad \bar{w} = \frac{1}{2}\sum_{n=0}^{\infty} w_{n+1}\sigma^n. \tag{70}$$

† For the σ calculus see e.g. [17]

The first term is found to be

$$\left.\frac{\Delta^{\frac{3}{2}}}{(2\sigma)^3}\right|_R^M = \frac{1}{4}(\nabla_a\nabla_b\nabla_c\Delta^{\frac{1}{2}} - 3\nabla_a\nabla_b\Delta^{\frac{1}{2}}{}_{;c} + 3\nabla_a\Delta^{\frac{1}{2}}{}_{;bc} - \Delta^{\frac{1}{2}}{}_{;abc})\sigma^a\sigma^b\sigma^c\frac{\ln(2M^2\sigma)}{(2\sigma)^3}. \tag{71}$$

Similarly the second and third terms can be computed:

$$\left.\frac{\Delta v_0\ln(2\sigma)}{(2\sigma)^2}\right|_R^M = \frac{1}{4}(\nabla_a v_0 - v_{0;a})\sigma^a\frac{\Delta\ln(2M^2\sigma)\ln\left(\frac{2\sigma}{M^2}\right)}{(2\sigma)^2} \tag{72}$$

$$\left.\frac{\Delta w_0}{(2\sigma)^2}\right|_R^M = \frac{1}{2}(\nabla_a w_0 - w_{0;a})\sigma^a\frac{\Delta\ln(2M^2\sigma)}{(2\sigma)^2}. \tag{73}$$

Without further knowledge about the Feynman propagator, the β-function can be evaluated to lowest order, leading to the result $\beta_2 = 3$ in this renormalization scheme. The calculation requires the use of the identity $\nabla_a\frac{\Delta\sigma^a}{(2\sigma)^2} = 2\pi^2\delta(p, p')$.

5. Conclusions

The elegant method of CPT is not only suited for the investigation of renormalizability but also for the performance of perturbative computations. The subtraction procedure on the test functions can be pulled back to the distributions yielding differential renormalization. Therefore it is possible to work in the standard integral kernel representation.

As the whole procedure is formulated in configuration space, it can be transferred to distributions on a manifold. This enables one to give compact expressions for the renormalization of quantum fields in curved spacetime, at least in the Euclidean case.

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References

[1] Blanchard P and Seneor R 1975 Green’s functions in theories with massless particles (in perturbation theory) *Ann. Inst. Henri Poincaré* A **XXIII** 147–209

[2] Bogoliubov N N and Shirkov D V 1976 *Introduction to the Theory of Quantized Fields* 3rd edn (New York: Wiley)

[3] Brunetti R and Fredenhagen K 1996 Interacting quantum fields in curved space: renormalizability of φ^4 *Operator Algebras and Quantum Field Theory* ed S Doplicher *et al* (Hong Kong: International) (Brunetti R and Fredenhagen K 1997 *Preprint* gr-qc/9701048)

[4] Brunetti R, Fredenhagen K and Köhler M 1996 The microlocal spectrum condition and wick polynomials of free fields on curved spacetimes *Commun. Math. Phys.* **180** 633–52

[5] Bunch T S 1981 BPHZ renormalization of $\lambda\phi^4$ field theory in curved spacetime *Ann. Phys., NY* **131** 118–48

[6] Comellas J, Haagensen P E and Latorre J I 1995 Differential renormalization for curved space and finite temperature *Int. J. Mod. Phys.* A **10** 2819–39

[7] DeWitt B S and Brehme 1960 Radiation damping in a gravitational field *Ann. Phys., NY* **9** 220–59

[8] Dosch H G and Müller V F 1975 Renormalization of quantum electrodynamics in an arbitrarily strong time independent external field *Fortschr. Phys.* **23** 661–89

[9] Dütsch M, Hurth T, Krahe F and Scharf G 1994 Causal construction of Yang-Mills theories. I *Nuovo Cimento* A **106** 1029–41

[10] Dütsch M, Hurth T, Krahe F and Scharf G 1994 Causal construction of Yang-Mills theories. II *Nuovo Cimento* A **107** 375–406

[11] Dütsch M, Hurth T, Krahe F and Scharf G 1995 Causal construction of Yang-Mills theories. III *Nuovo Cimento* A **108** 679–708
[12] Dütsch M, Hurth T and Scharf G 1995 Causal construction of Yang-Mills theories. IV. unitarity *Nuovo Cimento* A **108** 737–74
[13] Epstein H and Glaser V 1973 The role of locality in perturbation theory *Ann. Inst. Henri Poincare* A **XIX** 211–95
[14] Feynman R P, Leighton R B and Sands M 1964 *Lectures on physics* (Reading, MA: Addison-Wesley)
[15] Fredenhagen K 1996 *Renormierungstheorie (Lecture notes)*
[16] Freedman D Z, Johnson K and Latorre J I 1992 Differential regularization and renormalization: a new method of calculation in quantum field theory *Nucl. Phys.* B **371** 352–414
[17] Fulling S 1989 *Aspects of Quantum Field Theory in Curved Spacetime* (Cambridge: Cambridge University Press)
[18] Haagensen P E and Latorre J I 1992 Differential renormalization of massive quantum field theories *Phys. Lett.* B **283** 293–7
[19] Hörmander L 1990 *The analysis of Linear Partial Differential Operators I* 2nd edn (Berlin: Springer)
[20] Kay B S and Wald R M 1991 Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate killing horizon *Phys. Rep.* **207** 49–136
[21] Latorre J I, Manuel C and Vilasís-Cardona X 1994 Systematic differential renormalization to all orders *Ann. Phys., NY* **231** 149–73
[22] Lüscher M 1982 Dimensional regularisation in the presence of large background fields *Ann. Phys., NY* **142** 359–92
[23] Prange D 1997 Causal perturbation theory and differential renormalization *Preprint* hep-th/9710225
[24] Scharf G 1995 *Finite Quantum Electrodynamics* 2nd edn (Berlin: Springer)
[25] Smirnov V A and Zav'yalov O I 1993 On differential renormalization *Theor. Math. Phys.* **96** 974–81
[26] Steinmann O 1971 Perturbation expansions in axiomatic field theory (*Springer Lecture Notes in Physics vol 11*) (Berlin: Springer)
[27] Stora R 1993 Differential algebras in lagrangean field theory *ETH Lectures (Lecture Notes)*